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STABILIZATION OF A LINEAR STOCHASTIC SYSTEM SUBJECTED TO "WHITE NOISE"-TYPE PARAMETRIC DISTURBANCES

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Necessary and sufficient conditions of stability of a linear system of automatic control the parameters of which are subjected to the action of a white noise, are obtained. The stabilizing control is constructed according to the principle of linear feed-back, by measuring all coordinates of the phase vector of the system.

We consider a linear object described by the following system of differential equations:

$$dx/dt = Ax(t) + b\sigma + q\varphi \tag{1}$$

where A is a real constant $n \times n$ matrix, b and q are constant n-vectors and $x(t)$ denotes the phase vector of the object. The input φ is closed in the following manner:

$$\varphi = (r^*x)\xi \tag{2}$$

where ξ is white noise of unit spectral density and r is a constant n-vector. We require to close the input using the feed-back

$$\sigma = c^*x \tag{3}$$

in such a manner that the resulting system (1) — (3) of stochastic differential equations is stable (the solution of (1) — (3) is understood in the sense of [1]).

The stochastic stability has been defined by a number of authors (see e. g. [2, 3]). We shall have in mind the exponential mean square stability [1].

Definition [1]. The system (1) — (3) will be called exponentially stable in the mean square, if positive numbers N and ε exist such that for every $t \geqslant t_0$ and for any n-dimensional vector x_0 the following inequality holds:

$$M\,|x(t)|^2 \leqslant N\,|x_0|^2 \exp \varepsilon (t_0 - t)$$

where $x(t)$ is a solution of the system (1) — (3) with the initial condition $x(t_0) = x_0$, and M denotes the mathematical expectation.

The problem of stabilization is formulated as follows: to find a vector c for which the system (1) — (3) is exponentially stable in the quadratic mean. The aim of this paper is to find the effective algebraic conditions of existence of the stabilizing vector c.

The following notation will be used in formulating these conditions:

$$A_\lambda = \lambda I - A, \quad \delta(\lambda) = \det A_\lambda, \quad \chi(\lambda) = r^* A_\lambda^{-1} b, \quad \theta(\lambda) = \delta(\lambda)\,\chi(\lambda), \quad \Phi(\lambda) = \theta(\lambda)\theta(-\lambda). \tag{4}$$

The function $\chi(\lambda)$ is a transfer function of the linear object (1) from the input σ to the output $\psi = r^*x$, $\chi(\lambda)$ is a bilinear function, and $\theta(\lambda)$ is a polynomial of degree $k \leqslant n - 1$, where n is the order of the system (1).

We shall assume that the pair (A, b) is controllable, i. e. that the vectors $b, Ab, \ldots,$

$A^{n-1}b$ are linearly independent.

In this case [4] we can choose a vector c such that the characteristic polynomial of the matrix $P = A + bc^*$ has any specified roots. In particular, we can make the matrix P to satisfy the Hurwitz conditions with any set of the eigenvalues. This, however, will guarantee the stability of the system (1) — (3) only when the vector q of noise intensity is sufficiently small (in the norm). The necessary and sufficient conditions for the exponential stability of the system (1) — (3) have been given in a number of sources (see 1 — 3, 5, 6]). In the present case the most suitable criterion of stability is the one given in [3, 6]. The results of these papers imply that the system (1) — (3) will be exponentially stable if and only if the following inequality holds:

$$q^*Hq < 1 \tag{5}$$

Here H denotes the solution of the Liapunov matrix equation

$$HP + P^*H = -rr^* \quad (P = A + bc^*) \tag{6}$$

in which the matrix P. is defined by the expression in brackets. For this reason we can replace the question of stabilization of the system (1) — (3) by the question of existence of a vector c for which the inquality (5) holds; the answer to this is given by the following theorem.

Theorem 1. Let the matrix A and vector b in system (1) be such that the pair (A, b) is controllable. We denote by $\zeta(\lambda)$ a polynomial of degree $k \leqslant n - 1$ with the principal coefficient $\varkappa > 0$ satisfying the relation $\zeta(\lambda)\zeta(-\lambda) = \Phi(\lambda)$ and possessing no roots in the half-plane $\operatorname{Re}\lambda > 0$. Such a polynomial exists and is unique. Let the vector d and the matrix H be given by the relations

$$d^*Q(\lambda)b = \zeta(\lambda) \tag{7}$$

$$HA + A^*H = -rr^* + dd^*, \quad Hb = 0 \tag{8}$$

Then a real vector c stabilizing the system (1) — (3) will exist if and only if the inequality

$$q^*Hq < 1 \tag{9}$$

holds. The theorem will be proved below.

We note that the relations (8) form an overdefined system of equations. We shall see from the proof of the theorem that Eqs. (8) are always compatible, and this means that the matrix H is determined uniquely. In fact, if a matrix H satisfying the system (8) exists, then H satisfies the first equation of (8) containing any matrix $A' = A + bc^*$, where c is an arbitrary vector. This is due to the fact that $HA = HA'$.

Theorem 2. Let the relation $q = b$ hold in the system (1). Then the system (1) is stabilizable.

Theorem 3. Let the polynomial $\theta(\lambda)$ defined by the penultimate formula of (4) have no roots in the half-plane $\operatorname{Re}\lambda > 0$. Then the system (1) — (3) is stabilizable for any value of the vector q.

Theorem 3 admits the following interpretation. The transfer functions of the stationary linear objects with finite-dimensional phase space are bilinear functions. The object is called a minimum phase object when the numerator of this function is a Hurwitz polynomial. As we said before, $\chi(\lambda)$ can be regarded as a transfer function from the input σ in the output $\psi = r^*x$. From Theorem 3 it follows that if this transfer function has a minimum phase, the object perturbed by a white noise of intensity linearly dependent on the phase coordinates of the system can be stabilized at any noise level. We note that if the transfer function vanishes in the half-plane $\operatorname{Re}\lambda > 0$, the noise exists under which stabilization is not possible.

Example 1. Let us consider the following linear stochastic differential equation:

$$x^{(n)} + (a_{n-1} + r_{n-1}\xi)x^{(n-1)} + \ldots + (a_0 + r_0\xi)x = \sigma$$

where ξ is the white noise. A linear combination $\sigma = c_0 x + c_1 x' + \ldots + c_{n-1}x^{(n-1)}$ can always be chosen such that the trivial solution of this equation is exponentially stable in the mean square (the case $b = q$).

Example 2. Consider a system containing two integrable terms

$$x_1^{\cdot} = \sigma, \quad x_2^{\cdot} = kx_1 + \varphi\xi, \quad \varphi = r_1x_1 + r_2x_2$$

Let us determine the range of variation of the parameters r_1, r_2, k over which the above system can be stabilized. In this case the matrix A and the vectors b, q and r have the form

$$A = \begin{Vmatrix} 0 & 0 \\ k & 0 \end{Vmatrix}, \quad b = \begin{Vmatrix} 1 \\ 0 \end{Vmatrix}, \quad q = \begin{Vmatrix} 0 \\ 1 \end{Vmatrix}, \quad r = \begin{Vmatrix} r_1 \\ r_2 \end{Vmatrix}$$

It is clear that

$$\chi(\lambda) = \frac{1}{\lambda^2}(r_1\lambda + kr_2), \quad \theta(\lambda) = r_1\lambda + kr_2$$

When $kr_1r_2 \geqslant 0$, the transfer function $\chi(\lambda)$ has a minimum phase and the system (1) can be stabilized. Consider the case $kr_1r_2 < 0$. Then the polynomial $\theta(-\lambda)$ satisfies the Hurwitz condition and the following relation holds for $\zeta(\lambda)$: $\zeta(\lambda) = |r_1|\lambda + |kr_2|$. From this it follows that $d_1 = r_1$, $d_2 = |kr_2|/k$.

We have the following equation for the matrix H:

$$HA + A^*H = -rr^* + dd^*$$

and from this it follows that $h_{11} = h_{12} = h_{21} = 0$, $h_{22} = -2r_1r_2/k$.

The inequality $q^*Hq < 1$ yields the inequality $-2r_1r_2/k < 1$, and the latter represents the necessary and sufficient condition of stabilizability of the system in question.

The proof of the theorems formulated above is based on a number of auxiliary lemmas. Let us introduce the following notation:

$$Q(\lambda) = \delta(\lambda)A_\lambda^{-1}, \quad \Phi_\alpha(\lambda) = \Phi(\lambda) + \alpha^2\Pi(\lambda) \tag{10}$$
$$\Pi(\lambda) = \delta(\lambda)\,\delta(-\lambda) + b^TQ^T(-\lambda)\,Q(\lambda)\,b$$

where A and $\Phi(\lambda)$ are defined by the formulas (4). Clearly Φ and Φ_α are polynomials the degree of which does not exceed $2n$ where n is the order of the system (1). Their roots are situated symmetrically about the real and imaginary axes, and the roots lying on the imaginary axis are of even multiplicity. Therefore the polynomials Φ and Φ_α can be written in the form

$$\Phi(\lambda) = \zeta(-\lambda)\,\zeta(\lambda), \quad \Phi_\alpha(\lambda) = \zeta_\alpha(\lambda)\,\zeta_\alpha(-\lambda) \tag{11}$$

where ζ and ζ_α are polynomials with real coefficients, with no roots in the half-plane $\operatorname{Re}\lambda > 0$. Under this condition the factorization of the form shown in (11) is unique.

Lemma 1. The following relation holds:

$$\lim_{\alpha\to 0}\zeta_\alpha(\lambda) = \pm\zeta(\lambda)$$

Proof. A set of polynomials of degree $k \leqslant n$ forms a linear space of dimension

$n+1$. Let us introduce in this space the following norm:

$$\|\zeta\| = \left\{\int_0^1 |\zeta(i\omega)|^2 d\omega\right\}^{1/2}$$

The second relations of (10) and (11) together imply that

$$\|\zeta_\alpha\|^2 = \mu + \alpha^2\nu, \quad \mu = \int_0^1 \Phi(i\omega)\, d\omega, \quad \nu = \int_0^1 \Pi(i\omega)\, d\omega$$

The family of polynomials ζ_α can be regarded as a trajectory in a finite-dimensional linear space. The assertions made above imply that this trajectory is bounded. Let

$$\zeta_{\alpha_k} \to \zeta^* \quad \text{as } \alpha_k \to \beta, \quad \beta \in (0, 1)$$

Since the roots of the polynomials $\zeta_{\alpha_k}(\lambda)$ lie in the half-plane $\operatorname{Re}\lambda \leqslant 0$, so do the roots of the polynomials ζ^*. Moreover, the coefficient accompanying the highest degree terms in the polynomials ζ_α is equal to α, so that $\zeta^*(\lambda)$ is a polynomial satisfying the conditions of factorization given above. Consequently, $\zeta^*(\lambda) = \zeta_\beta(\lambda)$, i.e. when $\alpha \to \beta$, the trajectory ζ_α has a unique limit point represented by the polynomial $\zeta_\beta(\lambda)$, and this means that the trajectory ζ_α is continuous at the point $\alpha = \beta > 0$.

When $\alpha \to 0$, the positiveness of the coefficient accompanying the highest degree term in the polynomial $\zeta^*(\lambda)$ can no longer be guaranteed. Repeating the arguments, we find that when $\alpha \to 0$, the trajectory ζ_α can only have two limit elements represented by the polynomials ζ and $-\zeta$. Since for a bounded continuous trajectory in a finite-dimensional space the set of limit elements is either infinite or consists of a single element, therefore when $\alpha \to 0$ we can have either $\lim \zeta_\alpha = \zeta$, or $\lim \zeta_\alpha = -\zeta$, and this completes the proof of Lemma 1.

Lemma 2. Let $\rho = \inf_c q^*Hq$, where H is a solution of the matrix equation (6) with the Hurwitz matrix $P = A + bc^*$. The following relation holds:

$$\rho = \inf_{u\in U} I(u), \quad I(u) = \int_0^\infty |r^*x(t)|^2\, dt$$

where $x(t)$ is the solution of the equation

$$dx/dt = Ax(t) + bu(t), \quad x(0) = q \tag{12}$$

$$\left(\int_0^\infty |u(t)|^2\, dt < +\infty, \quad \int_0^\infty |x(t)|^2\, dt < +\infty\right)$$

and the set U of functions $u(t)$ is defined by the expression within the brackets.

Proof. Let H be a solution of $HP + P^*H = -rr^*$, where $P = A + bc^*$ is a Hurwitz matrix. It is evident that

$$q^*Hq = \int_0^\infty |r^*x(t)|^2\, dt$$

where $x(t)$ is a solution of (12), with the function $u(t) = c^*x(t)$.

Let us consider the functional

$$I_\alpha(u) = \int_0^\infty [|r^*x|^2 + \alpha^2|x|^2 + \alpha^2u^2]\, dt$$

The integrand function has been altered in order to make it a positive definite quadratic function of the arguments x and u. As we know (see [7]), $\min I_\alpha(u)$ exists and can be attained on the function $u(t)$ connected with the solution $x(t)$ of the system (12) by the relation $u = c_\alpha{}^* x(t)$, and the matrix $P_\alpha = A + bc_\alpha{}^*$ satisfies the Hurwitz conditions. As the result of all this, we have the following obvious relationships:

$$\inf I_\alpha(u)_{\alpha\to 0} \to \inf I(u), \quad \inf I_\alpha(u) \geqslant \rho, \quad \inf I(u) \leqslant \rho$$

and from this we have $\inf I(u) = \rho$.

From Lemma 2 it follows that the necessary and sufficient condition of existence of the vector c stabilizing the system (1) — (3) is, that the inequality

$$\rho = \inf I(u) < 1$$

where $I(u)$ is the functional defined in Lemma 2, holds. Since $\rho = \lim_{\alpha\to 0}\rho_\alpha$ where $\rho_\alpha = \min I_\alpha(u)$, the question of stabilizing the system (1) — (3) reduces to that of determining the quantity ρ_α. As was shown in [7], the quantity ρ_α can be attained on the function $u_\alpha = c_\alpha{}^* x(t)$, where $x(t)$ is a solution of (12), and the vector c_α is independent of the initial conditions.

In accordance with the procedure described in [8], the vector c_α' is uniquely determined from the linear relations introduced by the identity

$$\alpha c_\alpha{}^* Q(\lambda) b = \alpha\delta(\lambda) - (-1)^n \zeta_\alpha(\lambda) \tag{13}$$

where $\zeta_\alpha(\lambda)$ is a Hurwitz polynomial satisfying the condition of factorization

$$\zeta_\alpha(\lambda)\,\zeta_\alpha(-\lambda) = \Phi_\alpha(\lambda)$$

and $\delta(\lambda)$, $Q(\lambda)$, $\Phi_\alpha(\lambda)$ are given by the formulas (4) and (10).

We have the following relation for the vector $d_\alpha = \alpha c_\alpha$:

$$d_\alpha{}^* Q(\lambda)\, b = \alpha\delta(\lambda) - (-1)^n \zeta_\alpha(\lambda) \tag{14}$$

We note that the polynomial appearing in the right-hand side is of degree $k \leqslant n - 1$. As is shown in [8], the matrix $P_\alpha = A + bc_\alpha{}^*$ satisfies the Hurwitz conditions. Let us introduce the matrix H_α, satisfying the relation

$$H_\alpha P_\alpha + P_\alpha{}^* H_\alpha = -(rr^* + \alpha^2 I + \alpha^2 c_\alpha c_\alpha{}^*) \tag{15}$$

The matrix H_α satisfies, at the same time, the relations (see [8])

$$q^* H_\alpha q = \rho_\alpha, \quad H_\alpha b = -\alpha d_\alpha, \quad H_\alpha A + A^* H_\alpha = -rr^* - \alpha^2 I + d_\alpha d_\alpha{}^* \tag{16}$$

From Lemma 1 it follows that when $\alpha \to 0$ either $\zeta_\alpha \to \zeta$, or $\zeta_\alpha \to -\zeta$. Passing to the limit in the relations (14) — (16), we obtain the proof of Theorem 1.

The proof of Theorem 2 follows from the second relation of (8), and in this case we have $\rho = 0$.

Let us now prove Theorem 3. Let the polynomial $\theta(\lambda)$ have no roots in the half-plane $\operatorname{Re}\lambda > 0$. Then the polynomial $\zeta(\lambda)$ appearing in the statement of Theorem 1 coincides with the polynomial $\theta(\lambda)$. Therefore the vector d defined by (7) coincides with the vector r. From (8) it follows that $H = 0$, therefore $\rho = 0$ and the system (1) — (3) is stabilizable.

Note. The proof of Theorem 1 contains a method of constructing a stabilizing feedback of the form (3). If $\rho < 1$, then the inequality $\rho_\alpha < 1$ holds for a sufficiently small $\alpha > 0$. We shall show that the vector c_α which represents the solution of (13), is the stabilizing vector.

Since $P_\alpha = A + bc^*$ is a Hurwitz matrix, a unique solution of the matrix equation $HP_\alpha + P_\alpha^*H = -rr^*$ exists. As the matrix $G = H_\alpha - H$ satisfies the equation $GP_\alpha + P_\alpha^*G = -\alpha^2(I + c_\alpha c_\alpha^*)$, it follows that $G > 0$ and we have $q^*Hq < q^*H_\alpha q < 1$. As we have shown above, the condition $q^*Hq < 1$ represents the sufficient condition of stability of the system (1) — (3).

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